EXAMPLE OF THE GENERATION OF A SECONDARY STATIONARY OR PERIODIC FLOW WHEN THERE IS LOSS OF STABILITY OF THE LAMINAR FLOW OF A VISCOUS INCOMPRESSIBLE FLUID

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It is well-known that the solution of a stationary problem for the Navier-Stokes equations is unique 'in the small' (for a small Reynolds number, small mass forces, etc.). However, experiments and approximate calculations using Galerkin's method show that, in general, there is no uniqueness. For example, as experiment shows, a secondary stationary flow may arise with the loss of stability of a Couette flow between rotating cylinders. Up to now, however, the non-uniqueness has not been rigorously proved in a single case.

In the present paper the question of the bifurcation of stationary flows of a viscous incompressible fluid is considered. In section 1 the theory of bifurcation of solutions of operational equations, developed in [1] is shown to be applicable to this problem. Thus, the question of bifurcation of the Navier-Stokes equations reduces to the determination of the odd-multiple characteristic values which correspond to a linearized problem.

In section 2 a two-dimensional problem of the Navier-Stokes equations is considered with a periodicity condition on the stream function with respect to x, y corresponding to the periods $2\pi/\alpha_0, 2\pi$.

In [2] the stability of the solution of this problem $\psi_0 = -(\gamma/\nu)\cos y$ was investigated (in the usual linear formulation) and it was shown that stability always occurs for $\alpha_0 > 1$ and is lost for sufficiently small α_0, fixed (γ/ν) and small ν. A proof of the method of linearization is given in [4].

In section 2 of the present paper it is shown that for $\alpha_0 \geqslant 1$ the stationary solution ψ_0 is stable 'in the large' and unique (2.2), and that for any $\alpha_0 < 1$ and sufficiently large values of the parameter $\lambda = (\gamma/\nu^2)$ new stationary solutions branch off from the solution ψ_0. (Revising the result from [2] by the method of section 2 (cf. lemma 2.1) it is not difficult to show that the solution ψ_0 is unstable for $\lambda > \lambda(\alpha_0)$ where $\lambda(\alpha_0)$ is the point of bifurcation found in section 2 (2.3).)

A basic result on non-uniqueness is formulated in theorem 2, where the exact number of points of bifurcation are determined depending on α_0 and some conclusions about the spectrum are drawn. From theorem 2 theorem 3 easily follows in which an example of the generation of a time periodic flow when there is loss of stability of a stationary flow is given as well as an example of the generation of a conditionally periodic flow when there is loss of stability of a periodic flow.

1. On the bifurcation of stationary solutions of the Navier-Stokes equations.

1.1. *Reduction to an operational equation.* We shall consider the stationary problem of the Navier-Stokes equations in the bounded domain D with the boundary S

$$\nu\Delta v_i' = v_k' \frac{\partial v_i'}{\partial x_k} + \frac{\partial P}{\partial x_i} - F_i \quad (i=1,2,3), \quad \operatorname{div}\mathbf{v}' \equiv \frac{\partial v_i'}{\partial x_i} = 0, \quad \mathbf{v}'|_S = \mathbf{a} \tag{1.1}$$

where $\mathbf{F}$, $\mathbf{a}$ are given vectors; $\mathbf{v}'(x)$, $P(x)$ are the unknown velocity and pressure. We shall employ the usual convention of omitting the summation sign for the repeated index.

We shall assume that $\mathbf{F}$, $\mathbf{a}$ depend on a parameter γ and that for any γ $(-\infty < \gamma < \infty)$ the problem (1.1) admits a solution of the form

$$\mathbf{v}' = \gamma\mathbf{v}_0(x), \qquad P = P_0(x_0, \gamma) \tag{1.2}$$

where $\mathbf{v}_0$ no longer depends on γ. It is well known that for small γ this solution is unique. Later we shall be interested in solutions of problem (1.1) which are different from (1.2). We shall seek them in the form

$$\mathbf{v}' = \gamma\mathbf{v} + \gamma\mathbf{v}_0, \qquad P = (1/\nu\gamma)\, p + P_0 \tag{1.3}$$

For determining $\mathbf{v}$ and p we shall then obtain the problem

$$\Delta v_i = \lambda\left[v_{0k}\frac{\partial v_i}{\partial x_k} + v_k\frac{\partial v_{0i}}{\partial x_k} + v_k\frac{\partial v_i}{\partial x_k}\right] + \frac{\partial p}{\partial x_i}, \quad \operatorname{div} v \equiv \frac{\partial v_i}{\partial x_i} = 0, \quad v|_S = 0$$
$$(\lambda = \gamma/\nu) \tag{1.4}$$

Along with the problem (1.4) we shall consider the linearized problem which corresponds to it

$$\Delta u_i = \lambda\left[v_{0k}\frac{\partial u_i}{\partial x_k} + u_k\frac{\partial v_{0i}}{\partial x_k}\right] + \frac{\partial q}{\partial x_i}, \qquad \frac{\partial u_i}{\partial x_i} = 0, \qquad \mathbf{u}|_S = 0 \tag{1.5}$$

and the problem conjugate to (1.5)

$$\Delta w_i = -\lambda v_{0k}\left(\frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i}\right) + \frac{\partial Q}{\partial x_i}, \qquad \frac{\partial w_i}{\partial x_i} = 0, \qquad \mathbf{w}|_S = 0 \tag{1.6}$$

We shall reduce these three problems to equations with completely continuous operators. This is achieved essentially by inversion of the operators which correspond to $\lambda = 0$. We shall introduce a Hilbert space H_1 – a complete set of smooth solenoidal vectors which vanish near S, by the norm generated by the scalar products

$$(\mathbf{u}, \mathbf{w})_{H_1} = \int_D \frac{\partial \mathbf{u}}{\partial x_k} \frac{\partial \mathbf{w}}{\partial x_k} dx = \int_D \text{rot } \mathbf{u} \cdot \text{rot } \mathbf{w} \, dx \tag{1.7}$$

According to a theorem of imbedding [3], the inequality

$$\| \mathbf{u} \|_{L_p} = \left(\int_D | \mathbf{u} |^p dx \right)^{1/p} \leqslant c_p \| \mathbf{u} \|_{H_1} \qquad (1 \leqslant p \leqslant 6) \tag{1.8}$$

is correct for all $\mathbf{u} \in H_1$, where c_p depends only on the domain D and on p, but not on $\mathbf{u}$.

The vectors $\mathbf{v}, \mathbf{u}, \mathbf{w} \in H_1$, which differ from zero and satisfy the integral identities

$$(\mathbf{v}, \mathbf{\Phi})_{H_1} = -\lambda \int_D \left[v_{0k} \frac{\partial v_i}{\partial x_k} + v_k \frac{\partial v_{0i}}{\partial x_k} + v_k \frac{\partial v_i}{\partial x_k} \right] \Phi_i \, dx \tag{1.9}$$

$$(\mathbf{u}, \mathbf{\Phi})_{H_1} = -\lambda \int_D \left[v_{0k} \frac{\partial u_i}{\partial x_k} + u_k \frac{\partial v_{0i}}{\partial x_k} \right] \Phi_i \, dx \tag{1.10}$$

$$(\mathbf{w}, \mathbf{\Phi})_{H_1} = \lambda \int_D v_{0k} \left(\frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i} \right) \Phi_i \, dx \tag{1.11}$$

for all $\mathbf{\Phi} \in H_1$ and some λ, the so-called eigen-value, will be called the generalized eigen-vectors of the problems (1.4)-(1.6), respectively. From the results of [4] (cf. also [5]) it follows that for sufficiently smooth $\mathbf{F}$, S and $\mathbf{a}$ the generalized eigen-vectors together with some functions $p(x)$, $q(x)$, and $Q(x)$ will generate solutions of the problems (1.4)-(1.6) in the classical sense.

We shall now determine the operators K, A, A^* which act in H_1 by requiring that the integral identities

$$(K\mathbf{v}, \mathbf{\Phi})_{H_1} = -\int_D \left[v_{0k} \frac{\partial v_i}{\partial x_k} + v_k \frac{\partial v_{0i}}{\partial x_k} + v_k \frac{\partial v_i}{\partial x_k} \right] \Phi_i \, dx \tag{1.12}$$

$$(A\mathbf{u}, \mathbf{\Phi})_{H_1} = -\int_D \left(v_{0k} \frac{\partial u_i}{\partial x_k} + u_k \frac{\partial v_{0i}}{\partial x_k} \right) \Phi_i \, dx \tag{1.13}$$

$$(A^*\mathbf{w}, \mathbf{\Phi})_{H_1} = \int_D v_{0k} \left(\frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i} \right) \Phi_i \, dx \tag{1.14}$$

be satisfied for any $\mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{\Phi} \in H_1$.

Lemma 1.1. The operators K, A, A^* are completely continuous in H_1.

The proof for the operator K is given in [4]. For A and A^* it is exactly the same.

Lemma 1.2. The problems of determining the generalized eigen-vectors defined in (1.9)-(1.11) are equivalent to the corresponding operational equations

$$\mathbf{v} = \lambda K\mathbf{v}, \qquad \mathbf{u} = \lambda A\mathbf{u}, \qquad \mathbf{w} = \lambda A^*\mathbf{w} \tag{1.15}$$

The validity of this lemma follows immediately from the definitions of the generalized eigen-vectors and the operators K, A, A^*.

Lemma 1.3. The operator A is a Fréchet differential of the operator K at the point $\mathbf{v} = 0$.

Proof. It is necessary to show that

$$\lim \frac{\|K\mathbf{u} - A\mathbf{u}\|_{H_1}}{\|\mathbf{u}\|_{H_1}} = 0 \qquad \text{for} \quad \|\mathbf{u}\|_{H_1} \to 0 \tag{1.16}$$

From (1.12) and (1.13) we conclude that

$$(K\mathbf{u} - A\mathbf{u}, \mathbf{\Phi})_{H_1} = -\int\limits_D u_k \frac{\partial u_i}{\partial x_k} \Phi_i \, dx = \int\limits_D \mathbf{u} \times \operatorname{rot} \mathbf{u} \cdot \mathbf{\Phi} \, dx \tag{1.17}$$

We shall set $\mathbf{\Phi} = K\mathbf{u} - A\mathbf{u}$ in (1.17). Evaluating the right-hand side of (1.17) by means of the Hölder inequality with indices $p_1 = 4$, $p_2 = 2$, $p_3 = 4$ and applying the imbedding inequality (1.8), we obtain

$$\|K\mathbf{u} - A\mathbf{u}\|_{H_1}^2 \leqslant \|\mathbf{u}\|_{L_4} \cdot \|\operatorname{rot} \mathbf{u}\|_{L_2} \|K\mathbf{u} - A\mathbf{u}\|_{L_4} \leqslant c_4^2 \|\mathbf{u}\|_{H_1}^2 \cdot \|K\mathbf{u} - A\mathbf{u}\|_{H_1}$$

Thus

$$\|K\mathbf{u} - A\mathbf{u}\|_{H_1} \leqslant c_4^2 \|\mathbf{u}\|_{H_1}^2 \tag{1.18}$$

and hence (1.16) follows.

Lemma 1.4. The operator A^* is the conjugate of the operator A in H_1.

Proof. Let $\mathbf{w}$ and $\mathbf{\Phi}$ be arbitrary vectors from H_1. Integrating (1.14) by parts, we find

$$(A^* \mathbf{w}, \mathbf{\Phi})_{H_1} = -\int\limits_D w_i \left[v_{0k} \frac{\partial \Phi_i}{\partial x_k} + \Phi_k \frac{\partial v_{0i}}{\partial x_k} \right] dx = (\mathbf{w}, A\mathbf{\Phi})_{H_1} \tag{1.19}$$

which proves the lemma.

We can now apply the theory of bifurcation of solutions of non-linear operational equations [1] to the determination of stationary flows which are different from (1.2).

The real number λ_0 is called a point of bifurcation of the operator K, if for any $\epsilon, \delta > 0$ a characteristic number λ of the operator K can be found such that $|\lambda - \lambda_0| < \epsilon$ even though $\|\mathbf{v}\|_{H_1} < \delta$ for some eigen-vector $\mathbf{v}$ of the operator K which corresponds to this characteristic number. Only the characteristic numbers of its Fréchet differential at zero, the operator A, can be points of bifurcation of the operator K.

In the case under consideration a theorem of M.A. Krasnosel'skii [1] gives the following: let λ_0 be a characteristic number of the operator K having odd multiplicity. Then the number λ_0 is a point of bifurcation of the operator K; moreover, a continuous branch of eigen-vectors of the operator K corresponds to this point of bifurcation.

We shall explain the concepts applied here. Let λ_0 be a characteristic number of the operator A and $\mathbf{u}$ be any of the eigen-vectors corresponding to it. We shall consider the following problem:

$$\mathbf{u} = \lambda_0 A\mathbf{u}, \qquad \mathbf{u}_1 = \lambda_0 A\mathbf{u}_1 + \mathbf{u}_0, \ldots, \qquad u_r = \lambda_0 A\mathbf{u}_r + \mathbf{u}_{r-1}, \ldots \tag{1.20}$$

As is well known, the complete continuity of the operator A implies that only a finite number r of them are resolvable. In this connection r is called the rank of the eigen-vector $\mathbf{u}$.

If $r = 1$ we shall say that the eigen-vector $\mathbf{u}$ is simple. The vectors $\mathbf{u}_1$, $\mathbf{u}_2$, . . ., $\mathbf{u}_{r-1}$ are called adjoints to the eigen-vector $\mathbf{u}$. The linear envelope of all the eigen-vectors and adjoint vectors corresponding to the given characteristic number λ_0 is called the invariant subspace of the operator A which corresponds to the characteristic number λ_0.

The dimension n of this subspace is called the multiplicity of the characteristic number λ_0.

If $n = 1$, λ_0 is then called a simple characteristic number; if $n > 1$, λ_0 is called a multiple characteristic number.

Something can be learned about the spectrum of the operator K (i.e., about the totality of its characteristic values) with the help of the following consideration. For every λ obtained from the characteristic numbers of the operator A the index of the stationary point $\mathbf{v} = 0$ of the vector field $(I - K)\,\mathbf{v}$ is equal to unity in absolute magnitude and changes sign when λ passes through an odd-multiple characteristic number of the operator A. On the other hand, the calculation of the rotation of the vector field $I - K$ on spheres of large radius turns out well in many cases.

For example, in [4] it is shown to be equal to +1, if the vector flux $\mathbf{a}$ through every component of the boundary S is equal to zero. In this case we obtain (cf. [1]) that the interval between two characteristic numbers of the operator A, where the index of the stationary point $\mathbf{v} = 0$ is -1, entirely contains the spectrum of the operator K.

1.2. *On determining the multiplicity of a characteristic number.* It was shown above that to prove the existence of a point of bifurcation of the operator K it is necessary firstly to establish that the operator A has a real characteristic number and secondly to show that it is an odd multiple. But the operator is not self-adjoint and, in general, can not have real characteristic values. For example, if $\mathbf{v}_0(x)$ represents the motion of the fluid as a solid body, then real eigen-values do not exist. Moreover, in this case the stationary solution (1.2) is unique for any γ (these facts follow immdediately from (1.9) and (1.10) if it is assumed here that $\Phi = \mathbf{v}$, $\Phi = \mathbf{u}$ respectively, and if it is noted that the right-hand sides now vanish).

Sometimes it is convenient to consider the operator A on the complex envelope $\mathbf{H}_1$ of the space H_1 and to use the following simple lemma in establishing the reality of an eigen-number and the simplicity of the eigen-values.

Lemma 1.5. Let λ_0 be a characteristic number of a real (i.e., transformed real vectors in the system) linear operator A acting in $\mathbf{H}_1$. In order that λ_0 be real and have the rank 1, it is necessary and sufficient that to every eigen-vector $\mathbf{u}$ with characteristic number λ_0 there correspond at least one eigen-vector $\mathbf{w}$ of the conjugate operator A^* which has the same characteristic number and which is not orthogonal to $\mathbf{u}$

$$(\mathbf{u}, \mathbf{w})_{H_1} \neq 0 \tag{1.21}$$

Proof. For the proof we note that a necessary and sufficient condition for solving the equation

$$\mathbf{u}_1 = \lambda_0 A \mathbf{u}_1 + \mathbf{u} \tag{1.22}$$

is the satisfaction of the equality

$$(\mathbf{u}, \mathbf{w})_{H_1} = 0 \tag{1.23}$$

where $\mathbf{w}$ is any solution of the equation $\mathbf{w} = \lambda_0{}^* A^* \mathbf{w}$. If λ_0 is real, then $\lambda_0{}^* = \lambda_0$, and the need for condition (1.21) is demonstrated.

Now let (1.21) be satisfied. Then we have

$$(\mathbf{u}, \mathbf{w})_{H_1} = (\lambda_0 A \mathbf{u}, \mathbf{w})_{H_1} = \lambda_0 (\mathbf{u}, A^* \mathbf{w})_{H_1} = \lambda_0 \left(\mathbf{u}, \frac{1}{\lambda_0} \mathbf{w}\right)_{H_1} = \frac{\lambda_0}{\lambda_0{}^*} (\mathbf{u}, \mathbf{w})_{H_1} \tag{1.24}$$

and, since $(\mathbf{u}, \mathbf{w})_{H_1} \neq 0$, we obtain $\lambda_0 = \lambda_0{}^*$. From (1.21) the unsolvability of equation (1.22) now follows; hence, $\mathbf{u}$ is a simple eigen-vector. The lemma is proved.

It is important to note that approximate values of λ_0 and of the eigen-vectors can be used to check condition (1.21). This, incidentally, permits the existence of real positive eigen-values in the instability spectrum of Couette flow to be established; in [6] eigen-values with a positive real part were found.

Broad classes of linear operators with simple eigen-vectors exist, such as, for example, the self-adjoint operators. However, even for such operators the calculation of the multiplicity of an eigen-number is a difficult problem. As an example we shall consider the problem of the eigen-values for the Laplace operator in the rectangle $\{0 \leqslant x \leqslant \pi / \alpha, 0 \leqslant y \leqslant \pi\}$ with the condition that the function vanish on the boundary. The eigen-values are the numbers $\lambda_{kl} = -(\alpha^2 k^2 + l^2)$ $(k, l = 0, 1, 2, \ldots; \lambda_{kl} \neq 0)$. If α^2 is irrational, then they are all simple. For rational α^2 multiple eigen-values can also be found. Thus for $\alpha = 1$ all of the λ_{kl} with $k \neq l$ are multiples.

Sometimes the uniqueness of an eigen-vector can be obtained by narrowing down the space in which there is a solution. For example, let the problem

$$u'' = -\lambda(u + u^3), \qquad u(-\pi) = u(\pi), \qquad u'(-\pi) = u'(\pi) \quad (1.25)$$

be solved.

Turning to the operator d^2/dx^2, it can be reduced to an equation with a completely continuous operator. The problem on the eigen-values of this Fréchet differential is equivalent to the boundary value problem (1.25) with the u^3 term discarded. The characteristic values will be $\lambda_k = k^2$ and the eigen-functions $\varphi_{k_1} = \sin kx$, $\varphi_{k_2} = \cos kx$ correspond to each of them. Since the rank of λ_k is equal to 1, we obtain that its multiplicity is equal to x. However, if solutions of (1.25) which are odd with respect to x are sought, then the function ϕ_{k_2} is 'eliminated', the eigen-values λ_k become simple and each of them can be affirmed as a point of bifurcation.

2. An example of non-uniqueness of a stationary flow.

2.1. *Solutions of the two-dimensional stationary Navier-Stokes equations.* We shall consider the equations

$$\nu \triangle \mathbf{v} = (\mathbf{v} \cdot \nabla)\, \mathbf{v} + \nabla P - \mathbf{F}, \qquad \operatorname{div} \mathbf{v} = 0 \quad (2.1)$$

under the condition of periodicity of the velocity with respect to x, y with the periods $2\pi / \alpha_0$, 2π respectively. In addition, we shall require that the condition

$$\frac{\alpha_0}{4\pi^2} \int\limits_D \mathbf{v}(x, y)\, dx\, dy = \mathbf{b} \quad (2.2)$$

be satisfied, where D is the rectangle $\{|x| \leqslant \pi / \alpha_0, |y| \leqslant \pi\}$, and $\mathbf{b}$ is an unknown vector. We shall take $F_1 = -\gamma \sin y$, $F_2 = 0$, $\mathbf{b} = 0$. By introducing the stream function ψ the problem is reduced to determining a solution of the equation

$$\nu \triangle^2 \psi = \psi_y \triangle \psi_x - \psi_x \triangle \psi_y - \gamma \cos y \quad (2.3)$$

which is periodic with respect to x, y with the periods $2\pi / \alpha_0$, 2π. In order to fix the arbitrary additive in the determination of ψ, we shall further introduce the condition

$$\int\limits_D \psi\, dx\, dy = 0 \quad (2.4)$$

The problem (2.3) and (2.4) obviously has the solution

$$\psi_0 = -\gamma / \nu \cos y \quad (2.5)$$

The substitution

$$\psi = \gamma / \nu (\varphi_0 - \cos y)$$

reduces equation (2.3) to the form

$$\Delta^2\varphi_0 = \lambda [\varphi_{0y}\Delta\varphi_{0x} - \varphi_{0x}\Delta\varphi_{0y} + \sin y (\partial / \partial x) (\Delta\varphi_0 + \varphi_0)] \tag{2.6}$$

The corresponding linearized problem has the form

$$\Delta^2\varphi = \lambda \sin y (\partial / \partial x) (\Delta\varphi + \varphi) \qquad (\lambda = \gamma/\nu^2) \tag{2.7}$$

the adjoint problem is

$$\Delta^2\Phi = - \lambda (1 + \Delta) (\partial / \partial x) (\Phi \sin y) \tag{2.8}$$

The functions φ_0, φ, Φ must finally satisfy the condition of periodicity with respect to x, y and the condition (2.4).

The concepts of the previous section apply to the investigation of the problem (2.1)-(2.3). The only requirement is to define the space H_1 in a different way: instead of vanishing on the boundary it should be required that the vectors of H_1 now satisfy the condition of periodicity with respect to x, y as well as the homogeneous condition (2.2). However, it is more convenient to deal with stream functions here.

We shall define the Hilbert space H_2 as the closed set of smooth functions, which satisfy condition (2.4) and are periodic with periods $2\pi / \alpha_0$, 2π with respect to x, y and by the norm, generated by the product

$$(\psi_1, \psi_2)_{H_2} = \int_D \Delta\psi_1 \cdot \Delta\psi_2 \, dx \, dy$$

We shall define the operators L, B, B^* which act in H_2, requiring that the integral identities

$$(L\varphi, \Phi)_{H_2} = \int_D \Delta\varphi (\varphi_x\Phi_y - \varphi_y\Phi_x) \, dx \, dy - \int_D \sin y (\Delta\varphi + \varphi) \Phi_x \, dx \, dy$$

$$(B\varphi, \Phi)_{H_2} = - \int_D \sin y (\Delta\varphi + \varphi) \Phi_x \, dx \, dy$$

$$(B^*\varphi, \Phi)_{H_2} = \int_D (1 + \Delta) (\varphi \sin y) \Phi_x \, dx \, dy$$

be satisfied for any $\varphi, \Phi \in H_2$. As in section 1 (cf. lemmas 1.1-1.4), it is easily verified that L, B, B^* are completely continuous, B is a Fréchet differential of the operator L, B^* is adjoint to the operator B, and the problems (2.6), (2.7), and (2.8) are equivalent, respectively, to the operational equations

$$\varphi_0 = \lambda L\varphi_0 \tag{2.9}$$

$$\varphi = \lambda B\varphi \tag{2.10}$$

$$\Phi = \lambda B^*\Phi \tag{2.11}$$

2.2. *The condition of uniqueness and stability. Theorem* 2.1. Let $\alpha_0 \gg 1$. Then, whatever $\nu > 0$ and γ are, the stationary solution (2.5) is unique, and all solutions of the unsteady Navier-Stokes equation

$$\partial\Delta\psi \,/\, \partial t + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y - \nu\Delta^2\psi = -\gamma\cos y \tag{2.12}$$

which are periodic with respect to x, y with periods $2\pi \,/\, \alpha_0$, 2π tend toward it by the norm H_2 as $t \to \infty$.

Proof. We shall assume that $\psi = \psi_0 + \Phi$ in (2.12). The equation which $\Phi = \Phi\,(x,\, y,\, t)$, satisfies has the form

$$\frac{\partial\Delta\Phi}{\partial t} + \Phi_y\Delta\Phi_x - \Phi_x\Delta\Phi_y + \frac{\gamma}{\nu}\sin y\,\frac{\partial}{\partial x}\,(\Delta\Phi + \Phi) - \nu\Delta^2\Phi = 0 \tag{2.13}$$

Multiplying this equation $\Delta\Phi + \Phi$ and integrating over the rectangle D, we obtain

$$\frac{1}{2}\,\frac{d}{dt}\int\limits_D\,[(\Delta\Phi)^2 - (\nabla\Phi)^2]\,dxdy + \nu\int\limits_D\,[(\nabla\Delta\Phi)^2 - (\Delta\Phi)^2]\,dxdy = 0 \tag{2.14}$$

We shall expand Φ in a Fourier series in x, y

$$\Phi\,(x,\, y,\, t) = \sum_{k,l=-\infty}^{+\infty} c_{kl}\,\exp^{[i(\alpha_0 kx+ly)]} \tag{2.15}$$

We then obtain

$$\begin{aligned} J_1^2 &\equiv \int (\nabla\Phi)^2\,dxdy = \frac{4\pi^2}{\alpha_0}\sum_{k,l}\,(\alpha_0^2k^2 + l^2)\,|\,c_{kl}\,|^2 \\ J_2^2 &\equiv \int\limits_D (\Delta\Phi)^2\,dxdy = \frac{4\pi^2}{\alpha_0}\sum_{k,l}\,(\alpha_0^2k^2 + l^2)^2\,|\,c_{kl}\,|^2 \\ J_3^2 &= \int\limits_D (\nabla\Delta\Phi)^2\,dxdy = \frac{4\pi^2}{\alpha_0}\sum_{k,l}\,(\alpha_0^2k^2 + l^2)^3\,|\,c_{kl}\,|^2 \end{aligned} \tag{2.16}$$

The inequalities

$$J_1^2 \leqslant J_2^2 \leqslant J_3^2 \tag{2.17}$$

$$K_1^2 \leqslant K_2^2 \qquad (K_1^2 = J_2^2 - J_1^2,\; K_2^2 = J_3^2 - J_2^2) \tag{2.18}$$

$$K_1^2 \geqslant \frac{\alpha_0^2 - 1}{\alpha_0^2}\,J_2^2 \tag{2.19}$$

are easily derived from (2.16) for $\alpha_0 \geqslant 1$.

From (2.14) we derive the relation

$$\frac{1}{2}\frac{d}{dt}K_1^2 = -\nu K_2^2 \leqslant -\nu K_1^2, \qquad \frac{dK_1}{dt} \leqslant -\nu K_1 \tag{2.20}$$

from which follows

$$K_1(t) \leqslant e^{-\nu t} K_1(0) \tag{2.21}$$

For $\alpha_0 > 1$ it follows from (2.19), (2.21) that

$$J_2^2(t) = \int_D (\Delta\Phi)^2\, dxdy \to 0 \quad \text{as} \quad t \to \infty \tag{2.22}$$

and the theorem is proved in this case.

If $\alpha_0 = 1$, we then obtain

$$\Phi = \Phi_1 + \Phi_2, \qquad \begin{aligned} \Phi_1 &= c_{10}e^{ix} + c_{-1,0}e^{-ix} + c_{01}e^{iy} + c_{0,-1}e^{-iy} \\ \Phi_2 &= \sum_{k^2+l^2>1} c_{kl}\exp^{i(kx+ly)} \end{aligned} \tag{2.23}$$

We note that

$$K_1^2 = \frac{4\pi^2}{\alpha_0} \sum_{k^2+l^2>1} (k^2+l^2)(k^2+l^2-1)\,|c_{kl}|^2 \geqslant \frac{1}{2}\int_D (\Delta\Phi_2)^2\, dxdy = \mathbf{J}_2^2(t) \tag{2.24}$$

From (2.21) it follows that $\mathbf{J}_2(t) \to \infty\, 0$ as $t \to \infty$. But $J_2^2 = \mathbf{J}_2^2 + J_0^2$, where

$$J_0^2(t) = \int_D \Phi_1^2 dxdy \tag{2.25}$$

Hence, it remains to prove that $J_0(t) \to 0$ as $t \to \infty$. Substituting (2.23) into (2.13), multiplying the result by Φ_1 and integrating over D, we obtain

$$\frac{1}{2}\frac{d}{dt}J_0^2 + \nu J_0^2 = \int_D \Delta\Phi_2\,(\Phi_{2x}\Phi_{1y} - \Phi_{2y}\Phi_{1x})\, dx.dy \tag{2.26}$$

Using the simple bound

$$\Phi_{1x}^2 + \Phi_{1y}^2 \leqslant \frac{1}{2\pi^2} J_0^2 \tag{2.27}$$

the Buniakovskii inequality and the inequality (2.17) for Φ_2, we derive

$$\frac{1}{2}\frac{d}{dt}J_0^2 + \nu J_0^2 \leqslant \frac{1}{\pi\sqrt{2}} J_0\, \mathbf{J}_2^2$$

from (2.26) and taking (2.24) and (2.21) into account this gives the bound

$$J_0(t) \leqslant J_0(0)\, e^{-\nu t} + \frac{\sqrt{2}}{\pi} K_1^2(0)\, \frac{1 - e^{-\nu t}}{\nu}\, e^{-\nu t} \tag{2.28}$$

Hence $J_0(t) \to 0$ as $t \to \infty$. The theorem is proved.

2.3. *The spectrum of the linearized problem and bifurcation.* Every non-trivial solution of the problem (2.8) which corresponds to a given characteristic number λ is a linear combination of solutions of the form

$$\Phi = e^{i\alpha x} \sum_{n=-\infty}^{+\infty} (-1)^n c_n e^{iny} \tag{2.29}$$

where $\alpha = k\alpha_0$ (k is an integer) and the coefficients c_n satisfy the infinite system of linear algebraic equations ($n = 0, \pm 1, \pm 2, \ldots$)

$$a_n c_n + c_{n-1} - c_{n+1} = 0 \qquad \left(a_n = \frac{2}{\lambda} \frac{(n^2+\alpha^2)^2}{\alpha(\alpha^2-1+n^2)}\right) \tag{2.30}$$

We shall seek solutions of the system (2.30) such that $c_n \to 0$ as $|n| \to \infty$. From (2.30) it then follows that $|n|^k c_n \to 0$, also, whatever k is. Setting

$$\rho_n = \frac{c_n}{c_{n-1}}$$

we reduce (2.30) to the form

$$a_n + 1 / \rho_n = \rho_{n+1} \tag{2.31}$$

It follows from (2.31) that for any κ

$$\rho_k = -\frac{1}{a_k} + \frac{1}{a_{k+1}} + \ldots \tag{2.32}$$

The continued fraction (2.32) converge since $a_n \to +\infty$ as $n \to \infty$ (cf. [7]). From (2.31) there follows another expression for ρ_k

$$\rho_k = a_{k-1} + \frac{1}{a_{k-2}} + \frac{1}{a_{k-3}} + \ldots \tag{2.33}$$

Equating the right-hand sides of (2.32) and (2.33) to each other with $k = 1$, we obtain the following equation for determining the characteristic values λ, taking into account that $a_{-n} = a_n$:

$$-\frac{a_0}{2} = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots \equiv f(\lambda) \tag{2.34}$$

It is easy to verify that the right-hand sides of (2.32) and (2.33) coincide for all k provided that (2.34) is satisfied. If λ is a real root of equation (2.34), then the non-trivial solution of the system (2.30) with $|c_n| \to 0$ as $|n| \to \infty$ is unique to within a constant factor and is given by the formulas

$$\begin{aligned} c_0 &= 1, \; c_n = \rho_1\rho_2 \cdots \rho_n \quad (n > 0) \\ c_n &= (\rho_0\rho_{-1} \cdots \rho_{n+1})^{-1} \quad (n < 0) \end{aligned} \tag{2.35}$$

For definiteness, let $\lambda > 0$, and $\alpha > 0$. From what comes later (cf. lemma 2.3) it follows that α must be less than 1. Then $a_0 < 0$ and $a_k > 0$ for $k \neq 0$. Now from (2.32) for $k > 0$ and from (2.33) for $k < 0$ it follows that

$$\begin{aligned} |\rho_k| &< \frac{1}{a_k} \to 0 \qquad (k \to +\infty) \\ \rho_k &\geqslant a_{k-1} \to +\infty \qquad (k \to -\infty) \end{aligned} \tag{2.36}$$

With the help of (2.36) it is easy to verify that (2.35) gives a solution of the system (2.30) with $|c_n| \to 0$ as $|n| \to \infty$.

Equation (2.34) is derived in [2]; there it is also shown that $c_n \neq 0$ ($n = 0, \mp 1, \ldots$) for any solution with $|c_n| \to 0$ as $|n| \to \infty$ and, hence, the introduction of the quantities ρ_n is valid.

Lemma 2.1. Let $0 < \alpha < 1$. Equation (2.34) then has a positive root $\lambda = \lambda(\alpha)$ and, moreover, has only this one root.

Proof. We have

$$-\frac{a_0}{2} = \frac{1}{\lambda}\frac{\alpha^3}{1-\alpha^2} \tag{2.37}$$

For $f(\lambda)$ ($\lambda > 0$) the bound

$$f(\lambda) \leqslant \frac{1}{a_1} = \frac{\lambda}{2}\frac{\alpha^3}{(\alpha^2+1)^2} \tag{2.38}$$

is valid.

From (2.37) and (2.38) it is seen that $-{}^1/_2 a_0 > f(\lambda)$ for small $\lambda = 0$. We shall show that the converse inequality is valid for large λ. For this it is sufficient to establish that

$$\lambda f(\lambda) \to +\infty \quad \text{as} \quad \lambda \to +\infty$$

But for $f(\lambda)$ the bound

$$f(\lambda) > \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{2n}} = \frac{a_2 + a_4 + \ldots + a_{2n} + O(\lambda^{-2})}{1 + O(\lambda^{-2})} \tag{2.39}$$

obtains.

Setting

$$b_n = \lambda a_n = \frac{2(n^2+\alpha^2)^2}{\alpha(\alpha^2 - 1 + n^2)}$$

we derive from (2.39) that

$$\underline{\lim}\ \lambda f(\lambda) \geqslant \sum_{k=1}^{n} b_{2k} \quad \text{as} \quad \lambda \to \infty \tag{2.40}$$

The right-hand side of (2.40) tends to infinity as $n \to \infty$. Therefore, $\lambda f(\lambda) \to +\infty$ as $\lambda \to +\infty$.

Thus, equation (2.34) has a positive root. In order to prove its uniqueness, we shall show that $\lambda f(\lambda)$ is a monotonically increasing function. We have

$$\lambda f(\lambda) = \cfrac{1}{\cfrac{1}{\lambda} a_1 + \cfrac{1}{\lambda a_2 + \cfrac{1}{\cfrac{1}{\lambda} a_3 + \ldots}}} \tag{2.41}$$

When λ increases, the terms of this fraction with the odd numbers decrease and the terms with the even numbers do not change. Hence, $\lambda f(\lambda)$ increases. The lemma is proved.

Lemma 2.2. The positive root $\lambda = \lambda(\alpha)$ of equation (2.34) increases monotonically with α for $0 < \alpha < 1$.

Proof. We shall rewrite (2.34) in the form

$$\frac{1}{1-\alpha^2} = \cfrac{1}{\alpha^3 a_1 + \cfrac{\lambda}{\alpha^{-3} a_2 + \cfrac{1}{\alpha^3 a_3 + \ldots}}} \equiv \frac{1}{\alpha^3} \lambda f(\lambda) \tag{2.42}$$

The left-hand side of this equation is an increasing function of α. The proof will be complete if it is established that the right-hand side of (2.42) is a decreasing function of α. And this follows from the fact that the continued fraction (2.42) decreases when its odd terms increase and its even terms decrease and, in addition, from the fact that $\alpha^3 a_n$ increases with α

$$\frac{\partial}{\partial \alpha} \alpha^3 a_n = \frac{4\alpha(\alpha^2 + n^2)}{\lambda(\alpha^2 - 1 + n^2)^2} [2\alpha^4 + 3(n^2 - 1)\alpha^2 + n^2(n^2 - 1)] > 0 \qquad (n \geqslant 1)$$

and $\alpha^{-3} a_n$ decreases

$$\frac{\partial}{\partial \alpha} \alpha^{-3} a_n = -\frac{4(\alpha^2 + n^2)}{\lambda \alpha^5 (\alpha^2 - 1 + n^2)^2} [\alpha^4 + 3n^2\alpha^2 + 2n^2(n^2 - 1)] < 0 \qquad (n \geqslant 1)$$

Lemma 2.3. As $\alpha \to 1$ the positive root of equation (2.34) $\lambda = \lambda(\alpha) \to +\infty$. As $\alpha \to 0$ the root $\lambda(\alpha) \to \sqrt{2}$. For $\alpha \geqslant 1$ there are no real characteristic numbers.

Proof. By virtue of (2.34) and (2.38)

$$\frac{\alpha^3}{1-\alpha^2} = -\frac{a_0}{2}\lambda = \lambda f(\lambda) < \frac{1}{2}\lambda^2 \frac{\alpha^3}{(\alpha^2+1)^2}$$

and, hence,

$$\lambda^2 > \frac{2(\alpha^2+1)^2}{1-\alpha^2} \to \infty \qquad (\alpha^2 \to 1-0) \tag{2.43}$$

Further, from (2.34) it follows that

$$\frac{1}{\lambda^2} = -\frac{2}{b_0 b_1} - \eta, \qquad \eta = \frac{1}{\lambda b_1} \frac{1}{a_2 + \cfrac{1}{a_3} + \cdots} \qquad (b_n = \lambda a_n) \tag{2.44}$$

Passing to the limit as $\alpha \to 0$ in (2.44) and noting that

$$0 < \eta < \frac{1}{b_1 b_2} \to 0 \quad (\alpha \to 0), \qquad b_0 b_1 \to -1 \quad (\alpha \to 0)$$

we obtain

$$\lambda(\alpha) \to \sqrt{2} \qquad \text{as } \alpha \to 0$$

Finally, the non-existence of a root for $\alpha > 1$ follows immediately from (2.34) since in the case under consideration the left-hand and right-hand sides of the equation have different signs for any real λ. If $\alpha = 1$, then equation (2.34) has no meaning but it follows from (2.30) that $c_0 = 0$ and, after finding c_2, c_3, ... successively from (2.30) for $n = 1$, 2, ..., we obtain that $c_n \to +\infty$ as $n \to \infty$. Hence, there are no real characteristic numbers in this case also. The lemma is proved.

Lemma 2.4. Let $0 < \alpha_0 < 1$. Then equations (2.11) and (2.10) have exactly $[1/\alpha_0]$ positive* (and as many negative) characteristic numbers. Each of them has a multiplicity equal to 2.

Proof. Let $\alpha = \alpha_0 k < 1$; k is a positive integer. Then equation (2.11) has the eigen-function (2.29), where the c_n are defined by the equalities (2.35), and the characteristic number $\lambda = \lambda(\alpha_0 k)$ which corresponds to it is a positive root of equation (2.34) (cf. lemma 2.1).

The system (2.30) is invariant with respect to the substitutions $\alpha \to -\alpha$, $c_n \to (-1)^n c_n$. Therefore, the eigen-function obtained from this substitution into (2.29) will also correspond to the same characteristic number $\lambda = \lambda(\alpha_0 k)$. From lemmas 2.1 and 2.2 it follows that there are no other eigen-functions with the eigen-number $\lambda(\alpha_0 k)$.

We shall establish that the multiplicity of $\lambda(\alpha_0 k)$ is equal to 2, if we show that its rank is equal to 1. The real eigen-functions have the form $\Phi = c_1\Phi_1 + c_2\Phi_2$, in which

$$\Phi_1 = f(y)\, e^{i\alpha x} + f^*(y)\, e^{-i\alpha x}, \qquad \Phi_2 = i\,[f(y)\, e^{i\alpha x} - f^*(y)\, e^{-i\alpha x}] \tag{2.45}$$

where

$$f(y) = \sum_{n=-\infty}^{+\infty} (-1)^n c_n e^{iny} \tag{2.46}$$

* $[1/\alpha_0]$ denotes the number of positive integers less than $1/\alpha_0$.

It is immediately verified that from the eigen-functions of problem (2.7) or equation (2.10) there will be

$$\varphi_1 = g(y)\, e^{i\alpha x} + g^*(y)\, e^{-i\alpha x}, \qquad \varphi_2 = i\,[g(y)\, e^{i\alpha x} - g^*(y)\, e^{-i\alpha x}] \quad (2.47)$$

where

$$g(y) = \sum_{n=-\infty}^{+\infty} d_n e^{iny}, \qquad d_n = \frac{-c_n}{\alpha^2 + n^2 - 1}$$

We shall note one more relation needed for what follows. Multiplying (2.7) by $\Delta\varphi + \varphi$ and integrating over D, we obtain

$$\int_D (\nabla\Delta\varphi_k)^2\, dxdy - \int_D (\Delta\varphi_k)^2\, dxdy = 0 \qquad (k = 1, 2) \quad (2.48)$$

Taking (2.47) into account, we rewrite the equality (2.48) in the form

$$\sum_{n=-\infty}^{+\infty} (\alpha^2 + n^2)^2 (\alpha^2 + n^2 - 1)\, d_n^2 = 0 \quad (2.49)$$

We shall now calculate the quantities $(\varphi_i, \Phi_k)_{H_1}$. We have

$$(\varphi_1, \Phi_1)_{H_2} = \int_D \Delta\varphi_1 \cdot \Delta\Phi_1\, dxdy = \frac{8\pi^2}{\alpha_0} \sum_{n=-\infty}^{+\infty} (-1)^{n-1} (n^2 + \alpha^2)^2 (\alpha^2 + n^2 - 1)\, d_n^2 \quad (2.50)$$

or, taking (2.49) into account,

$$(\varphi_1, \Phi_1)_{H_2} = \frac{32\pi^2}{\alpha_0} \sum_{n=1,3,5,\ldots} (n^2 + \alpha^2)(\alpha^2 + n^2 - 1)\, d_n^2 > 0 \quad (2.51)$$

Later, we shall convince ourselves directly that

$$(\varphi_1, \Phi_2)_{H_2} = (\varphi_2, \Phi_1)_{H_2} = 0 \quad (2.52)$$

Thus, if $\varphi = c_1\varphi_1 + c_2\varphi_2$ (c_1, c_2 are real constants) is any eigen-function of problem (2.9) and $\Phi = c_1\Phi_1 + c_2\Phi_2$, then

$$(\varphi, \Phi)_{H_2} = c_1^2 (\varphi_1, \Phi_1) + c_2^2 (\varphi_2, \Phi_2) > 0 \quad (2.53)$$

According to lemma 1.5 this means that the rank of the eigen-number $\lambda = \lambda(\alpha_0 k)$ is equal to 1. Thus, the multiplicity of this characteristic number is equal to 2. The lemma is proved.

Lemma 2.5. The rotation of the vector field $\Omega_1\varphi = (I - \lambda L)\varphi$ [cf. (2.9)] on a sphere of sufficiently large radius with center at 0 is equal to + 1.

Proof. It is sufficient to prove [1] that the deformation

$$\Omega_t\varphi = (I - t\lambda L)\,\varphi \ (0 \leqslant t \leqslant 1)$$

brings about the homotopy of the vector field Ω_1 and the unit field $\Omega_0\varphi = \varphi$. To do this it is necessary to establish only that all of the zeros of the field Ω_t lie in a sphere of known radius (which is independent of t). But if $\Omega_t\varphi = 0$, then, according to the definition of the operator L, we have

$$(\varphi, \Phi)_{H_2} = t\lambda \int_D \Delta\varphi \, (\varphi_x\Phi_y - \varphi_y\Phi_x)\, dx\, dy - t\lambda \int_D \sin y\, (\Delta\varphi + \varphi)\, \Phi_x dx\, dy \tag{2.54}$$

for any $\Phi \in H_2$. Setting

$$\varphi = \psi + \cos y, \Phi = \psi_t$$

in (2.54), we find

$$\|\psi\|_{H_2}^2 = \int_D \Delta\psi \cos y\, dx\, dy \tag{2.55}$$

from which, applying the Buniakovskii inequality and the inequality $\|\varphi\| \leqslant \|\psi\| + \|\cos y\|$, we conclude that

$$\|\varphi\|_{H_2} \leqslant 2\|\cos y\|_{H_2} = 2\pi \sqrt{2/\alpha_0} \tag{2.56}$$

This completes the proof of the lemma.

Theorem 2.2. Let $0 < \alpha_0 < 1$. Then there exist exactly $m = [1/\alpha_0]$ positive numbers $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m$, which are points of bifurcation of equation (2.9). To each of them there corresponds a continuous branch of eigen-functions of equation (2.9) which are non-trivial solutions of problem (2.6). The numbers $-\lambda_1, -\lambda_2, \ldots, -\lambda_m$ are also points of bifurcation. The spectrum of equation (2.9) contains all of the intervals $(\lambda_1, \lambda_2), (\lambda_3, \lambda_4), \ldots, (\lambda_{2k-1}, \lambda_{2k})$, and also the intervals which are symmetric to them the negative semi-axis. If m is odd, then it also contains the intervals $(-\infty, -\lambda_m)$, (λ_m, ∞).

Proof. Let H_2° be a subspace of H_2, consisting of functions which satisfy the condition $\psi(-x, -y) = \psi(x, y)$. It is immediately verified that the operators L, B, B^* act in H_2°. In addition, the spectrum of operator B considered in H_2° coincides with the spectrum of operator B on all of H_2 and consists of the numbers $\mp\lambda(\alpha_0), \mp\lambda(2\alpha_0), \ldots, \mp\lambda(m\alpha_0)$, where $m = [1/\alpha_0]$ (cf. lemma 2.4). To a characteristic number $\lambda_k = \lambda(\alpha_0 k)$ $(k = 1, 2, \ldots, m)$ there corresponds only one eigen-function Φ_1, defined in (2.45). The rank of λ_k according to lemma 2.4 is equal to 1. Thus, all of the characteristic numbers $\mp\lambda_1, \mp\lambda_2, \ldots, \mp\lambda_m$ are simple. Hence, according to a theorem of M.A. Krasnosel'skii [1] stated in section 1, all of them are points of bifurcation and to each of them corresponds a continuous branch of eigen-functions of the operator L.

If λ belongs to one of the intervals indicated in the condition of the theorem, then the index of the fixed point $\phi_0 = 0$ of the operator L is equal to -1. And therefore, according to lemma 2.5, the rotation of the vector field $(I - L)\phi$ on large spheres is equal to $+1$ for such λ that correspond to non-trivial solutions of equation (2.9). The theorem is proved.

2.3. An example of the generation of a periodic regime. We shall consider the problem (2.1), (2.2) with the previously used quantity $\mathbf{F} = (-\gamma \sin y, 0)$ and with $\mathbf{b} = (U, 0)$. This problem has the stationary solution

$$v_{01} = \gamma / \nu \sin y + U, \qquad v_{02} = 0, \quad P_0 = 0 \tag{2.57}$$

The velocity vector $\mathbf{v}_0$ has the stream function

$$\psi_0' = -\frac{\gamma}{\nu}\cos y + U_y$$

If we assume in (2.12) $\psi = \psi_0' + \Phi$, we obtain the periodicity conditions for $\Phi(x, y, t)$ with respect to x and y with the periods $2\pi / \alpha_0$, 2π and the equation

$$\frac{\partial \Delta\Phi}{\partial t} + \Phi_y \Delta\Phi_x - \Phi_x \Delta\Phi_y + U\Delta\Phi_x + \frac{\gamma}{\nu}\sin y \frac{\partial}{\partial x}(\Delta\Phi + \Phi) - \nu\Delta^2\Phi = 0 \tag{2.58}$$

Theorem 2.3. For all values of λ for which equation (2.6) has non-trivial solutions, equation (2.58) has non-trivial solutions which are periodic with respect to time.

Proof. For some λ let there be a non-trivial solution $\varphi_0(x, y)$ of equation (2.6) and let it have a period $2\pi / \alpha$ with respect to x (α is a multiple of α_0).

Then, it is easy to be convinced that

$$\Phi_0 = \varphi_0(x - Ut, y) \tag{2.59}$$

is a solution of problem (2.58) which is periodic with respect to time with period $\omega = 2\pi / \alpha U$. The theorem is proved.

We note that the flow in (2.59) is nothing other than stationary flow in a coordinate system which moves along the x-axis with constant velocity U.

We shall now make the coordinate system also move along the y-axis with velocity V. It is not difficult to be convinced that the flow with the stream function

$$\psi = -\frac{\gamma}{\nu}\cos(y - Vt) + Uy - Vx + \varphi_0(x - Ut, y - Vt) \tag{2.60}$$

presents an example of a conditionally periodic flow arising when there is loss of stability of a flow which is periodic with respect to time with the stream function

$$\psi_0'' = -\frac{\gamma}{\nu}\cos(y - Vt) + Uy - Vx$$

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